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A BRIEF ACCOUNT OF H. GRASSMANN'S GEOMETRICAL THEORIES.

By MR. ALEXANDER ZIWET, Washington, D. C.

[CONTINUED FROM VOL. II, PAGE 11.]

THE SYSTEM OF THE PLANE.

XVIII. A plane is generated by the *simple* motion of a straight line. Now, in general, a straight line in moving may change either its position (translation), or its direction (rotation), or both position and direction (twist), thus generating a cylinder, a cone, or a scroll, respectively. The scroll degenerates into a plane only when the twist reduces to mere translation or rotation. Cylindrical and conical surfaces reduce to planes if the direction of the motion is guided by a straight line. The plane then is generated either by the translation of a line one of whose points moves along another straight line, or by a rotating line which in all its positions meets another straight line.

Just as a line has two opposite directions corresponding to the two possible directions of a point in describing it, so a plane presents two opposite *sides* corresponding to the two possible directions of motion of the generating line: a rotation, for instance, which from one side of the plane appears as right-handed or clockwise, appears as left-handed or counter-clockwise if looked at from the other side of the plane.

In the case of the plane, as in that of the straight line, the chief characteristic of the generating motion is its *simplicity*, that is, the property of being completely determined by two positions of the generating element.

XIX. If A, B, C are three points, any other point X in their plane may be expressed in the form

$$X = \alpha A + \beta B + \gamma C, \text{ where } \alpha + \beta + \gamma = 1.$$

Indeed, if the line CX meet AB in C', this point C' may be expressed in the

form $C' = \alpha' A + \beta' B$, where $\alpha' + \beta' = 1$; and then X may be determined through C and C', thus :—

$$X = \gamma' C' + \gamma C, \text{ where } \gamma + \gamma' = 1.$$

Hence, substituting, $$X = \alpha'\gamma' A + \beta'\gamma' B + \gamma C,$$

where $$\alpha'\gamma' + \beta'\gamma' + \gamma = \alpha' - \alpha'\gamma + \beta' - \beta'\gamma + \gamma = 1 - \gamma + \gamma = 1.$$

To find, by construction, the point $X = \alpha A + \beta B + \gamma C$, where $\alpha + \beta + \gamma = 1$, we may externally multiply the expression by any one of the given points, say A:

$$AX = \beta AB + \gamma AC,$$

which may be written (see Art. 16)

$$A(X - A) = \beta A(B - A) + \gamma A(C - A),$$

or $$X - A = \beta(B - A) + \gamma(C - A),$$

an expression for the vector to the point X.

XX. Let now e_1, e_2, e_3 be three unit-points; then, as appears from the preceding article, any point αe in the same plane can be expressed in the form

$$\alpha e = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \text{ where } \alpha_1 + \alpha_2 + \alpha_3 = \alpha;$$

and, *vice versa*, this expression always represents a point in the plane $e_1 e_2 e_3$ unless the co-efficient α of e vanishes. In this latter case we have $\alpha = \alpha_1 + \alpha_2 + \alpha_3 = 0$, hence

$$\alpha e = \alpha_1(e_1 - e_3) + \alpha_2(e_2 - e_3),$$

i. e. a *vector*, or, as we may also say, a point at infinity.* It is evident that any vector in the plane $e_1 e_2 e_3$ and hence any of its points at infinity, may be expressed in the form $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$, where $\alpha_1 + \alpha_2 + \alpha_3 = 0$.

Hence, in general, *a homogeneous linear function of three points represents a point or a vector in the plane of the three points according as the sum of their co-efficients is different from or equal to zero.* On the other hand, any magnitude of the first degree (multiple point or vector) contained in the plane may be expressed as a linear homogeneous function of any three points of the plane. The plane is, therefore, *a system of the third degree.* Any four magnitudes of the first degree in such a system are connected by a homogeneous linear equation. Indeed, let

*The addition and subtraction of vectors is here supposed as known, as it is practically not different from what is to be found in all text-books on quaternions. Indeed, neither Hamilton nor Grassmann professes to have originated the so-called *geometrical* addition of lines, which before their time had been suggested first by the theory of imaginaries and later by mechanical considerations.

$$a = a_1e_1 + a_2e_2 + a_3e_3, \quad c = \gamma_1e_1 + \gamma_2e_2 + \gamma_3e_3,$$
$$b = \beta_1e_1 + \beta_2e_2 + \beta_3e_3, \quad d = \delta_1e_1 + \delta_2e_2 + \delta_3e_3$$

be the four magnitudes in the plane $e_1e_2e_3$. We can eliminate e_1, e_2, e_3 from the four equations and obtain the relation

$$\begin{vmatrix} a & a_1 & a_2 & a_3 \\ b & \beta_1 & \beta_2 & \beta_3 \\ c & \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = 0,$$

This derivation of a segment from two parallel segments is strictly analogous to the derivation of a point from two given points, in the system of the line, by means of the formula $A = a_0e_0 + a_2e_2$. In both cases, the quotient $a_2 : a_0$ indicates the ratio in which the distance between the fixed elements is divided. The particular value -1 of this ratio requires special attention. Just as $e_2 - e_0$ was interpreted as representing the "step" from e_0 to e_2, or the amount of motion necessary to move the point e_0 in the simplest way into the position e_2, so here the difference $e_2e_3 - e_0e_1$ of two equal and parallel segments may be regarded as the space passed through by the segment e_0e_1 if moved in the simplest way into the position e_2e_3; that is, this difference may be said to represent the *parallelogram* $e_0e_1e_3e_2$.

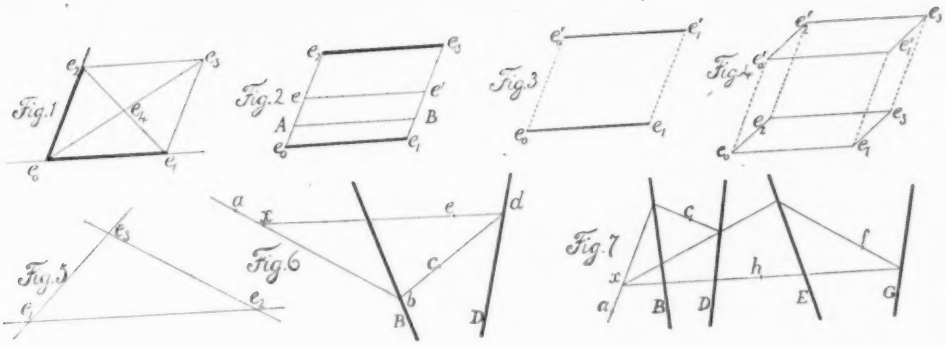
Fig.1
Fig.2
Fig.3
Fig.4
Fig.5
Fig.6
Fig.7

$$a = \alpha_1\epsilon_1 + \alpha_2\epsilon_2 + \alpha_3\epsilon_3, \quad c = \gamma_1\epsilon_1 + \gamma_2\epsilon_2 + \gamma_3\epsilon_3,$$
$$b = \beta_1\epsilon_1 + \beta_2\epsilon_2 + \beta_3\epsilon_3, \quad d = \delta_1\epsilon_1 + \delta_2\epsilon_2 + \delta_3\epsilon_3$$

be the four magnitudes in the plane $\epsilon_1\epsilon_2\epsilon_3$. We can eliminate ϵ_1, ϵ_2, ϵ_3 from the four equations and obtain the relation

$$\begin{vmatrix} a & \alpha_1 & \alpha_2 & \alpha_3 \\ b & \beta_1 & \beta_2 & \beta_3 \\ c & \gamma_1 & \gamma_2 & \gamma_3 \\ d & \delta_1 & \delta_2 & \delta_3 \end{vmatrix} = 0,$$

which is homogeneous and linear in a, b, c, d, any one of which may be expressed linearly by the three others.

XXI. The addition and substraction of *segments* in a plane is effected precisely as is the composition of forces in mechanics. We have (Fig. 1)

$$\epsilon_0\epsilon_1 + \epsilon_0\epsilon_2 = \epsilon_0(\epsilon_1 + \epsilon_2),$$

or, since $\epsilon_1 + \epsilon_2 = 2\epsilon_4$, and $2\epsilon_0\epsilon_4 = \epsilon_0\epsilon_3$,

$$\epsilon_0\epsilon_1 + \epsilon_0\epsilon_2 = \epsilon_0\epsilon_3.$$

This construction fails in the case of two parallel segments. We then have (Fig. 2)

$$\epsilon_0\epsilon_1 + \epsilon_2\epsilon_3 = \epsilon_0(\epsilon_1 - \epsilon_0) + \epsilon_2(\epsilon_3 - \epsilon_2) = (\epsilon_0 + \epsilon_2)(\epsilon_1 - \epsilon_0)$$
$$= 2\epsilon(\epsilon_1 - \epsilon_0) = 2\epsilon(\epsilon' - \epsilon) = 2\epsilon\epsilon',$$

where ϵ, ϵ' are the middle points respectively of $\epsilon_0\epsilon_2$ and $\epsilon_1\epsilon_3$.

Similarly, if

$$A = \alpha_0\epsilon_0 + \alpha_2\epsilon_2, \quad B = \alpha_0\epsilon_1 + \alpha_2\epsilon_3, \quad \therefore B - A = \epsilon_1 - \epsilon_0 = \epsilon_3 - \epsilon_2,$$

we have $A(B - A) = \alpha_0\epsilon_0(\epsilon_1 - \epsilon_0) + \alpha_2\epsilon_2(\epsilon_3 - \epsilon_2) = \alpha_0\epsilon_0\epsilon_1 + \alpha_2\epsilon_2\epsilon_3$.

Hence *any segment may be expressed as a sum of multiples of any two segments equal and parallel to it.*

This derivation of a segment from two parallel segments is strictly analogous to the derivation of a point from two given points, in the system of the line, by means of the formula $A = \alpha_0\epsilon_0 + \alpha_2\epsilon_2$. In both cases, the quotient $\alpha_2 : \alpha_0$ indicates the ratio in which the distance between the fixed elements is divided. The particular value -1 of this ratio requires special attention. Just as $\epsilon_2 - \epsilon_0$ was interpreted as representing the "step" from ϵ_0 to ϵ_2, or the amount of motion necessary to move the point ϵ_0 in the simplest way into the position ϵ_2, so here the difference $\epsilon_2\epsilon_3 - \epsilon_0\epsilon_1$ of two equal and parallel segments may be regarded as the space passed through by the segment $\epsilon_0\epsilon_1$ if moved in the simplest way into the position $\epsilon_2\epsilon_3$; that is, this difference may be said to represent the *parallelogram* $\epsilon_0\epsilon_1\epsilon_3\epsilon_2$.

This parallelogram is affected with sign, since the generating motion can be performed in two opposite directions. It changes its sign whenever the two segments are interchanged, and also if both segments are reversed, conditions which are in accordance with our notation as a difference:

$$e_2e_3 - e_0e_1 = -(e_0e_1 - e_2e_3) = -(e_3e_2 - e_1e_0) = e_1e_0 - e_3e_2.$$

XXII. *The external product of two magnitudes of the first degree* a and b, viz:—

$$ab = (a_1e_1 + a_2e_2 + a_3e_3)(\beta_1e_1 + \beta_2e_2 + \beta_3e_3)$$

$$= \begin{vmatrix} a_2 & a_3 \\ \beta_2 & \beta_3 \end{vmatrix} e_2e_3 + \begin{vmatrix} a_3 & a_1 \\ \beta_3 & \beta_1 \end{vmatrix} e_3e_1 + \begin{vmatrix} a_1 & a_2 \\ \beta_1 & \beta_2 \end{vmatrix} e_1e_2,$$

is a sum of multiple segments, and hence either a segment or a parallelogram.

Indeed if a and b are both points, we may take $a = a_1e_1$, $b = a_2e_2$; hence $ab = a_1a_2 \,.\, e_1e_2$, a segment.

If one of the factors is a point while the other is a vector, the latter may always be assumed to pass through the point, and we may take $a = a_1e_1$, $b = a_2(e_2 - e_1)$; hence $ab = a_1a_2e_1(e_2 - e_1) = a_1a_2 \,.\, e_1e_2$, a segment.

If, however, both a and b be vectors so that we have

$$a = a_1(e_1 - e_0), \quad b = a_2(e_2 - e_0),$$

we obtain

$$ab = a_1a_2[e_1(e_2 - e_0) - e_0e_2],$$

or, since $e_1(e_2 - e_0) = e_1e_3$ represents a segment passing through e_2 and equal in length and direction to the vector $e_2 - e_0$ (see Art. 16),

$$ab = a_1a_2(e_1e_3 - e_0e_2),$$

a parallelogram.

Segments and parallelograms may be called the *magnitudes of the second degree* in the system of the plane. The general expression of such a magnitude is

$$a = a_1e_0e_1 + a_2e_2e_3, \quad \text{or } a = a_1\varepsilon_1 + a_2\varepsilon_2,$$

where $\varepsilon_1 = e_0e_1$ and $\varepsilon_2 = e_2e_3$, which are any two segments in the plane, may be regarded as *units of the second degree*.

The relations between magnitudes of the second degree in the plane are analogous to those between magnitudes of the first degree in the system of the line (compare Arts. 12 and 13).

XXIII. The *external product* $e_1e_2e_3$ *of three unit-points* is called a *plane segment* (*Flächentheil*). It is a magnitude of the third degree, and we interpret it as representing a portion of the plane $e_1e_2e_3$ equal in area to twice the triangle contained between the three points. This plane segment $e_1e_2e_3$ is distinguished from the parallelogram $e_2e_3 - e_0e_1$ (where $e_1 - e_0 = e_3 - e_2$) in a way similar to that in which the linear segment e_0e_1 is distinguished from the vector $e_1 - e_0$. The seg-

ment e_0e_1, being confined to the line determined by the points e_0 and e_1, is a localized vector; the plane segment $e_1e_2e_3$ is a localized parallelogram, confined to the plane determined by the points e_1, e_2, e_3. While two parallelograms of equal area and sign are equal whenever their planes are parallel, two plane segments of the same area and sign are considered equal only if they lie in the same plane.

The vector $e_1 - e_0$ may be regarded as the difference of position of two points, that is, as a scalar quantity, having direction, but without any definite position in space. This quantity $e_1 - e_0$ if applied as co-efficient to a point e_0' (Fig. 3) forms a segment $e_0'(e_1 - e_0) = e_0'(e_1' - e_0') = e_0'e_1'$, or an extensive magnitude of one dimension, the numerical value, direction, and sign of which are the same as those of the vector, while its position is fixed through the point e_0'.

Similarly the parallelogram $e_2e_3 - e_0e_1 = \varepsilon_2 - \varepsilon_1$, was above defined as the difference of position of two parallel unit-segments; it is a scalar quantity possessing the common "aspect" of a system of parallel planes. If it be applied as co-efficient to a point e_0', we find (Fig. 4)

$$e_0'(e_2e_3 - e_0e_1) = e_0'(e_2'e_3' - e_0'e_1') = e_0'e_2'e_3',$$

a plane segment, i. e. an extensive magnitude of two dimensions.

XXIV. *The external product of three magnitudes of the first degree*

$$a = \alpha_1e_1 + \alpha_2e_2 + \alpha_3e_3,$$
$$b = \beta_1e_1 + \beta_2e_2 + \beta_3e_3,$$
$$c = \gamma_1e_1 + \gamma_2e_2 + \gamma_3e_3,$$

if formed according to the ordinary rules of algebra, is a sum of twenty-seven terms of the form $\alpha_i\beta_k\gamma_l e_ie_ke_l$. To reduce it to a simpler expression it becomes necessary to define the operation $(e_ie_k)e_l$, i. e. to establish a rule for multiplying products. We shall assume the associative law of algebra to hold in this case, i. e. we assume

$$(e_ie_k)e_l = e_i(e_ke_l) = e_ie_ke_l.$$

This law in connection with the two fundamental laws of external multiplication,

$$e_ie_k = -e_ke_i, \quad e_ie_i = 0,$$

will enable us to reduce all those products $e_ie_ke_l$ in the above sum which do not vanish to either of the forms $\pm e_1e_2e_3$, and we readily find

$$abc = \Sigma \pm \alpha_i\beta\gamma \cdot e_1e_2e_3 = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} e_1e_2e_3.$$

Hence *the external product of three magnitudes of the first degree is a multiple plane segment.*

It vanishes only when $\Sigma \pm a_i\beta\gamma = 0$, i. e. when the three magnitudes a, b, c are not independent of one another.

If a, b, c are all three vectors, they are always connected by such a relation between their co-efficients. Indeed, we then have

$$a_1 + a_2 + a_3 = 0, \quad \beta_1 + \beta_2 + \beta_3 = 0, \quad \gamma_1 + \gamma_2 + \gamma_3 = 0;$$

$$\therefore \begin{vmatrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1 + a_2 + a_3 & a_2 & a_3 \\ \beta_1 + \beta_2 + \beta_3 & \beta_2 & \beta_3 \\ \gamma_1 + \gamma_2 + \gamma_3 & \gamma_2 & \gamma_3 \end{vmatrix} = \begin{vmatrix} 0 & a_2 & a_3 \\ 0 & \beta_2 & \beta_3 \\ 0 & \gamma_2 & \gamma_3 \end{vmatrix} = 0.$$

The product of three complanar vectors always vanishes.

If a, b, c are not all three vectors, a relation between their co-efficients of the form $\Sigma \pm a_i\beta\gamma = 0$ indicates that the three magnitudes lie in the same straight line.

XXV. Just as, in the system of the line, by assuming a linear segment $e_1e_2 = 1$, we were enabled to express any other segment of the same line as a scalar quantity, so in the plane we may put $e_1e_2e_3 = 1$ and may then express any magnitude of the third degree in the same plane as a scalar, the plane segment $e_1e_2e_3$ being the geometrical unit for these magnitudes.

Again, remembering the rule that in an external product the first factor is the multiplicand and the second the multiplier, we call e_2e_3 the *complement* of e_1, in symbols $e_2e_3 = \mid e_1$; and this is really nothing but another way of stating the relation $e_1e_2e_3 = 1$, just as in ordinary algebra the expression $a = \frac{1}{b}$ is equivalent to the relation $ab = 1$.

We have thus:—

$$e_2e_3 = \mid e_1, \text{ since } e_1 \,.\, e_2e_3 = 1 \text{ (read: } e_1 \text{ multiplied by } e_2e_3 \text{ equals } 1),$$
$$e_3e_1 = \mid e_2, \text{ since } e_2 \,.\, e_3e_1 = -e_2e_1e_3 = +e_1e_2e_3 = 1,$$
$$e_1e_2 = \mid e_3, \text{ since } e_3 \,.\, e_1e_2 = -e_1e_3e_2 = +e_1e_2e_3 = 1.$$

Also:
$$e_1 = \mid e_2e_3, \text{ since } e_2e_3 \,.\, e_1 = -e_2e_1e_3 = +e_1e_2e_3 = 1,$$
$$e_2 = \mid e_3e_1, \text{ since } e_3e_1 \,.\, e_2 = -e_1e_3e_2 = +e_1e_2e_3 = 1,$$
$$e_3 = \mid e_1e_2, \text{ since } e_1e_2 \,.\, e_3 = 1.$$

Finally we may write

$$1 = \mid e_1e_2e_3, \text{ since } e_1e_2e_3 \,.\, 1 = e_1e_2e_3 = 1,$$
$$e_1e_2e_3 = \mid 1, \text{ since } 1 \,.\, e_1e_2e_3 = 1.$$

It follows at once from these formulæ that

$$\mid e_1 = \parallel e_2e_3 = e_2e_3,$$

i.e. *the complement of the complement of a magnitude is the original magnitude.* It

is to be observed that this rule is different from what we found in the system of the line where from $e_1e_2 = 1$ we deduced $e_2 = | e_1$ and $\| e_1 = -e_1$ (see Art. 17).

XXVI. The introduction of this idea of the complement of a factor is of special value as furnishing an interpretation, within the system of the plane, of products involving more than three unit-factors. Such products could otherwise be interpreted only as magnitudes of a degree higher than the third and would therefore not find any place in the system of the plane.

Thus, for instance, the product of two linear segments in the same plane, $e_1e_2 \,.\, e_2e_3$, may be written $| e_3 \,.\, | e_1$; and if in analogy with the law of ordinary algebra, $a \,.\, b = \dfrac{1}{\frac{1}{a} \,.\, \frac{1}{b}}$, we assume for our multiplication the analogous law, $a \,.\, b = | (| a \,.\, | b)$, we obtain

$$e_1e_2 \,.\, e_2e_3 = | (| e_1e_2 \,.\, | e_2e_3) = | (e_3 \,.\, e_1) = | e_3e_1 = e_2,$$

or, since $e_1e_2e_3 = 1$,

$$e_1e_2 \,.\, e_2e_3 = e_1e_2e_3 \,.\, e_2,$$

i. e. *the product of two linear segments in the same plane is their point of intersection.*

Similarly we find (Fig. 5)
$$e_3e_1 \,.\, e_1e_2 = e_1e_2e_3 \,.\, e_1,$$
$$e_1e_2 \,.\, e_2e_3 = e_1e_2e_3 \,.\, e_2,$$
$$e_2e_3 \,.\, e_3e_1 = e_1e_2e_3 \,.\, e_3;$$

multiplying these three equations we obtain

$$(e_2e_3 \,.\, e_3e_1 \,.\, e_1e_2)^2 = (e_1e_2e_3)^3,$$

or
$$e_2e_3 \,.\, e_3e_1 \,.\, e_1e_2 = (e_1e_2e_3)^2,$$

i. e. the product of three segments forming a triangle is equal to the square of the double area of this triangle. This product will therefore vanish only when the area of the triangle is equal to zero, i. e. when the three lines e_2e_3, e_3e_1, e_1e_2 pass through the same point. And, *vice versa*, whenever three such lines pass through the same point, we have the relation $e_2e_3 \,.\, e_3e_1 \,.\, e_1e_2 = 0$.

To extend these rules of multiplication to magnitudes of the first degree in general, we have only to define the complement of such a magnitude

$$a = a_1e_1 + a_2e_2 + a_3e_3;$$

and this is done by the formula

$$| a = a_1 | e_1 + a_2 | e_2 + a_3 | e_3.$$

It follows that if

$$a = a_1e_1 + a_2e_2 + a_3e_3, \quad b = {}_1\beta_1e_1 + {}_1\beta_2e_2 + {}_1\beta_3e_3,$$

$$\therefore ab = \begin{vmatrix} a_2 & a_3 \\ {}_1\beta_2 & {}_1\beta_3 \end{vmatrix} e_2e_3 + \begin{vmatrix} a_3 & a_1 \\ {}_1\beta_3 & {}_1\beta_1 \end{vmatrix} e_3e_1 + \begin{vmatrix} a_1 & a_2 \\ {}_1\beta_1 & {}_1\beta_2 \end{vmatrix} e_1e_2,$$

then $\quad | a = a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2, \quad | b = {}_1\beta_1e_2e_3 + {}_1\beta_2e_3e_1 + {}_1\beta_3e_1e_2,$

and $\quad | a \,.\, | b = \begin{vmatrix} a_2 & a_3 \\ {}_1\beta_2 & {}_1\beta_3 \end{vmatrix} e_1 + \begin{vmatrix} a_3 & a_1 \\ {}_1\beta_3 & {}_1\beta_1 \end{vmatrix} e_2 + \begin{vmatrix} a_1 & a_2 \\ {}_1\beta_1 & {}_1\beta_2 \end{vmatrix} e_3, \quad \therefore | (ab) = | a \,.\, | b.$

XXVII. Any magnitude in the system of the plane may be expressed by means of three fundamental-unit-points e_1, e_2, e_3; the double area of the triangle contained between these points being assumed as the geometrical unit for extensive magnitudes of two dimensions. The product of any two magnitudes of the first degree in e_1, e_2, e_3 is a magnitude of the second degree; the product of three magnitudes of the first degree, or of one magnitude of the first and one of the second, is a magnitude of the third degree. The degree thus being increased by multiplication, the product is called by Grassmann a *progressive* product.

When, however, the sum of the degrees of the factors exceeds three (i. e. the degree of the system of the plane), the degree of the resulting magnitude is less than the degree of the original magnitudes. The product of two linear segments, for instance, is their point of intersection. In this case Grassmann calls the product *regressive*.

In many geometrical investigations the value of the scalar coefficients which make the point an extensive magnitude of the first degree, and the line a linear segment of definite magnitude, are immaterial and may be disregarded. In this case we may say simply that the product of two points is the line connecting them; the product of two lines is their point of intersection; the product of a line and a point is a scalar which becomes zero when the point lies on the line. Any construction by means of the ruler alone, that is, requiring nothing but intersecting of lines and joining of points, may be expressed in the form of a continued product. Thus, if small letters be used to designate points and capitals be used to designate lines, the product *abCdEf* expresses the following construction: Join the points *a* and *b* by a line *ab*; the intersection of this line with the line *C* gives a point *abC* which joined to the point *d* determines a line *abCd* cutting the line *E* in a point *abCdE*; the line which joins this point to the point *f* is represented by *abCdEf*. Grassmann calls these expressions *planimetric* products.

XXVIII. If the resulting line were subject to the condition of passing through a given point *g*, we might express this simply by writing $abCdEfg = 0$; and the condition that a resulting point should lie in a given line would be

expressed in a similar way by equating a planimetric product to zero. If one of the points or lines used in the construction were not given, such an equation would evidently represent the geometrical locus of the point, or the envelope of the line, represented by the product.

Thus, for instance, we obtain the equation of a conic section by expressing the condition that the opposite sides of a hexagon one of whose vertices is variable intersect in three points situated in a straight line. Let a, b, c, d, e, x be any six points, of which the first five are given; the condition that x is a point of the conic determined by a, b, c, d, e, may be expressed by the equation

$$(ab \,.\, dc)\,(bc \,.\, ex)\,(cd \,.\, xa) = 0.$$

Newton's "organic" description of conics leads to another form of this equation. Let the three sides of a triangle xbd turn about three fixed points a, c, e, while two of its vertices, b and d, move along fixed lines B, D; then the third vertex x will describe the conic whose equation is

$$axBcDxe = 0.$$

The equation shows that a, e, BD, acD, ecB are points of the curve; for the substitution of these symbols for x makes the equation vanish identically.

A valuable investigation of certain properties of the "mystic hexagram" by means of Grassmann's methods was given by F. Caspary, in Crelle's *Journal*, Vol. XCII (1882), pp. 123–144.

XXIX. The most important property of such an equation is that it indicates, by the number of times the variable appears in it, the order or class of the locus or envelope it represents. For a rigorous proof of this proposition I must refer to the works of Grassmann and Schlegel. I can here only briefly indicate the application of the method to higher curves.

The equation $axBcDxEfGxh = 0$, which contains the variable point three times, represents a curve of the third order. The following linear construction of this curve may be directly read off from the equation: Let x be the common vertex of two contiguous triangles whose bases and not-common sides turn about the fixed points c, f, a, h, respectively, while the end-points of their bases move in the fixed lines B, D, E, G; then x describes a curve of the third order.

It is to be noticed that this is the first linear geometrical construction of the general cubic ever given. It was proposed by Grassmann in a paper published in Crelle's *Journal*, Vol. XXXI (1846), pp. 111–132.

The generalization of this method presents no difficulty. Indeed, let n triangles have a common vertex, the angles at this vertex being contiguous; then, if all the other vertices move in fixed straight lines, while the bases and the two

extreme sides turn about fixed points, the common vertex describes a curve of the n^{th} order, whose equation is

$$axB_1c_1D_1xB_2c_2D_2xB_3 \ . \ . \ . \ cx = 0.$$

Grassmann has also successfully applied his methods to the theory of projective ranges and pencils, and to the theory of harmonic poles and polars. In both cases was he led to noteworthy generalizations.

It is, however, in the science of rational mechanics that the application of Grassmann's theories and methods, in combination with those of Sir Wm. R. Hamilton, will probably prove of the greatest importance. As a valuable step in this direction, the reader might well be referred to the *Vector Analysis* of Professor J. Willard Gibbs (New Haven, 1881–1884), were it not for the fact that the words "not published" which appear on the title-page would seem to exclude that work from general circulation.

THE ATTRACTION OF A RIGHT CIRCULAR CYLINDER ON A PARTICLE.

By Chas. H. Kummell, Washington, D. C.

I propose to give here a more elaborate treatment of this problem, which is one of the illustrations in my paper, read before the Mathematical Section of the Philosophical Society of Washington entitled: *Can the Attraction of a Finite Mass be Infinite?*

Let m = the mass of attracted particle,

ϑ = density of attracting cylinder,

r = radius of attracting cylinder,

h = height of attracting cylinder,

d = distance of attracted particle from axis of cylinder,

e = elevation of attracted particle above base of cylinder.

Assume the vertical line through the attracted particle for the axis of cylindrical co-ordinates in which ρ = horizontal radius vector, v = horizontal angle with central radius vector, z = elevation above (or depression below) attracted particle; then, since

$$\sqrt{(\rho^2 + z^2)} = \text{distance of any mass element from particle}$$

and $\vartheta\rho d\rho dv dz =$ mass element,

we have $m\vartheta\dfrac{\rho d\rho dv dz}{\rho^2+z^2} =$ attraction of a mass element on the particle

and $m\vartheta\dfrac{\rho d\rho dv dz}{\rho^2+z^2}\cdot\dfrac{\rho\cos v}{\sqrt{(\rho^2+z^2)}} =$ component of this along central radius vector.

The sum of these components is the whole attraction; hence

$$A = m\vartheta\int_{d-r}^{d+r}\rho^2 d\rho\int_{-v}^{v}dv\cos v\int_{-e}^{h-e}\frac{dz}{\sqrt{(\rho^2+z^2)^3}},\text{ where }\cos v=\frac{\rho^2+d^2-r^2}{2\rho d},$$

$$= 2\,m\vartheta\int_{d-r}^{d+r}\rho^2 d\rho\,.\,\sin v\,.\,\frac{1}{\rho^2}\left(\frac{h-e}{\sqrt{[\rho^2+(h-e)^2]}}+\frac{e}{\sqrt{(\rho^2+e^2)}}\right),\text{ where}$$

$$\sin v=\frac{1}{2\rho d}\sqrt{[(d+r)^2-\rho^2]\,[\rho^2-(d-r)^2]},$$

$$= m\frac{\vartheta}{d}\int_{d-r}^{d+r}\frac{d\rho}{\rho}\sqrt{[(d+r)^2-\rho^2]\,[\rho^2-(d-r)^2]}\left(\frac{h-e}{\sqrt{[\rho^2+(h-e)^2]}}+\frac{e}{\sqrt{(\rho^2+e^2)}}\right). \tag{1}$$

The first term evidently expresses the attraction of the portion of the cylinder above the particle, which we shall denote A_{h-e}, and the second term that of the portion below the particle $= A_e$, and by simply replacing $h-e$ by e we obtain the second from the first and have $A = A_{h-e} + A_e$. To simplify, place in A_e

$$\begin{aligned} a_e &= \sqrt{[(d+r)^2+e^2]},\\ b_e &= \sqrt{[(d-r)^2+e^2]},\\ c_e &= \sqrt{(a_e^2-b_e^2)} = 2\sqrt{(rd)}; \end{aligned} \tag{2}$$

hence in A_{h-e}

$$\begin{aligned} a_{h-e} &= \sqrt{[(d+r)^2+(h-e)^2]},\\ b_{h-e} &= \sqrt{[(d-r)^2+(h-e)^2]},\\ c_{h-e} &= c_e = 2\sqrt{(rd)}; \end{aligned}$$

then
$$A_e = m\frac{\vartheta e}{d}\int_{d+r}^{d-r}\frac{d\rho}{\rho}\sqrt{\frac{[a_e^2-(\rho^2+e^2)]\,[(\rho^2+e^2)-b_e^2]}{\rho^2+e^2}}. \tag{3}$$

Let
$$\tan\varphi = \frac{a_e}{b_e}\sqrt{\frac{\rho^2+e^2-b_e^2}{a_e^2-\rho^2-e^2}},$$

and the new limits are $\lrcorner$ (quadrant) and o, and putting

$$\lrcorner\varphi = \sqrt{\left(1 - \frac{c_e^2}{a_e^2}\sin^2\varphi\right)}$$

$$= 1\sqrt{(a_e^2\cos^2\varphi + b_e^2\sin^2\varphi)}\left\{= \text{the elliptic } \lrcorner \text{ function to the modulus } \frac{c_e}{a_e}\right\},$$

we have also $$1\sqrt{(\rho^2 + e^2)} = \frac{b_e}{\lrcorner\varphi},$$

$$1\sqrt{[a_e^2 - (\rho^2 + e^2)]\,[(\rho^2 + e^2) - b_e^2]} = \frac{a_e b_e}{\lrcorner\varphi^2}\sqrt{\left(\lrcorner\varphi^2 - \frac{b_e^2}{a_e^2}\right)(1 - \lrcorner\varphi^2)},$$

$$\rho d\rho = \frac{b_e^2 c_e^2 \sin\varphi\cos\varphi d\varphi}{a_e^2\lrcorner\varphi^4} = \frac{b_e^2 c_e^2 d\varphi}{a_e^2\lrcorner\varphi^4}\cdot\frac{a_e^2}{c_e^2}\sqrt{(1 - \lrcorner\varphi^2)\left(\lrcorner\varphi^2 - \frac{b_e^2}{a_e^2}\right)},$$

$$\frac{d\rho}{\rho} = \frac{b_e^2 d\varphi\sqrt{(1 - \lrcorner\varphi^2)\left(\lrcorner\varphi^2 - \frac{b_e^2}{a_e^2}\right)}}{\left(\frac{b_e^2}{\lrcorner\varphi^2} - e^2\right)\lrcorner\varphi^4};$$

hence

$$A_e = m\frac{\vartheta e}{d}\int_0^{\lrcorner} \frac{d\rho\sqrt{(1 - \lrcorner\varphi^2)\left(\lrcorner\varphi^2 - \frac{b_e^2}{a_e^2}\right)}}{\left(1 - \frac{e^2}{b_e^2}\lrcorner\varphi^2\right)\lrcorner\varphi^2}\cdot\frac{\lrcorner\varphi}{b_e}\cdot\frac{a_e b_e}{\lrcorner\varphi^2}\sqrt{(1 - \lrcorner\varphi^2)\left(\lrcorner\varphi^2 - \frac{b_e^2}{a_e^2}\right)}$$

$$= m\vartheta a_e\frac{e}{d}\int_0^{\lrcorner} d\rho\,\frac{(1 - \lrcorner\varphi^2)\left(\lrcorner\varphi^2 - \frac{b_e^2}{a_e^2}\right)}{\left(1 - \frac{e^2}{b_e^2}\lrcorner\varphi^2\right)\lrcorner\varphi^3}$$

$$= -m\vartheta\frac{b_e^2 e}{a_e d}\int_0^{\lrcorner}\frac{d\rho}{\lrcorner\varphi^3} + m\vartheta\frac{b_e^2 a_e}{cd}\int_0^{\lrcorner}\frac{d\varphi}{\lrcorner\varphi} - m\vartheta\frac{(d^2 - r^2)^2}{a_e cd}\int_0^{\lrcorner}\frac{d\rho}{\left(1 - \frac{e^2}{b_e^2}\lrcorner\varphi^2\right)\lrcorner\varphi}$$

or, since $$\int_0^{\circ}\frac{d\varphi}{\lrcorner\varphi^3} = \frac{a_e^2}{b_e^2}\left(\int_0^{\circ} d\varphi\lrcorner\varphi - \frac{c_e^2\sin\varphi\cos\varphi}{a_e^2\lrcorner\varphi}\right)$$

$$A_e = m\vartheta\frac{b_e^2 a_e}{cd}\left\{\int_0^{\lrcorner}\frac{d\varphi}{\lrcorner\varphi} - \frac{e^2}{b_e^2}\int_0^{\lrcorner} d\varphi\lrcorner\varphi - \frac{(d + r)^2}{a_e^2}\int_0^{\lrcorner}\frac{d\varphi}{\left[1 + \frac{c_e^2}{a_e^2}\,\frac{e}{(d - r)^2}\sin^2\varphi\right]\lrcorner\varphi}\right\}$$

$$= \vartheta m \frac{b_e^2 a_e}{cd}\left[F - \frac{c^2}{b_e^2}E - \frac{(d+r)^2}{a_e^2}\Pi\left(\frac{c_e^2}{a_e^2}\cdot\frac{c^2}{(d-r)^2}\right)\right], \qquad (4)$$

where, by Legendre's notation, F, E, and Π are quadrantal elliptic integrals of the first, second, and third species respectively. The modulus is $\frac{c_e}{a_e}$ and the parameter $\frac{c_e^2}{a_e^2}\cdot\frac{c^2}{(d-r)^2}$.

Because the integral of the third species is quadrantal it may be expressed (as was first shown by Legendre) in terms of integrals of the first and second species, and I might apply his formula of reduction at once. I prefer, however, to make the reduction by means of Jacobi's theorem for exchanging amplitude and parameter (in Jacobi's sense), making use also of a new system of notation which I have used with advantage for many years.* In this notation I denote the modulus by γ and the complementary modulus by β (instead of k, k' by Jacobi). Taking the integral of the first species,

$$u = \int_0^{\varphi} \frac{d\varphi}{\sqrt{(1-\gamma^2\sin^2\varphi)}} = \int_0^{\varphi} \frac{d\varphi}{\Delta\varphi},$$

I write

$$u = \varphi_\gamma. \qquad (5)$$

If $\gamma = 0$, we have $u = \varphi_0 = \varphi$, and the integral is identical with its amplitude, and for any value of the modulus it may be looked upon (as the notation implies) as a modified amplitude.

Of (5) I take the inverse thus:—

$$u_{-\gamma} = \varphi. \qquad (6)$$

It should be noted here that u represents an integral and $u_{-\gamma}$ an angle of which trigonometric functions may be taken. Thus we have $\sin u_{-\gamma}$, $\cos u_{-\gamma}$, $\tan u_{-\gamma}$, and in addition the Δ function $\Delta u_{-\gamma} = \sqrt{(1-\gamma^2\sin^2 u_{-\gamma})}$, which are Jacobi's elliptic functions, and which he denoted as sin am u, cos am u, tan am u, Δ am u, a notation much complained of for its clumsiness. Gudermann proposed the notation sn u, cn u, tn u, dn u. This is short and convenient to write, but is apt to create the false impression that these quantities are no longer *ordinary* trigonometric functions. Besides, neither Jacobi's nor Gudermann's notation can make the distinction between different moduli conveniently.

*I have explained and used this notation in a paper on the quadric transformation read before the Mathematical Section of the Philosophical Society of Washington.

To express the quadrantal integral we might write

$$\left(\frac{\pi}{2}\right)_\gamma = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\Delta\varphi}.$$

But it is evidently desirable for this purpose to have a single symbol for $\frac{\pi}{2}$ and I have adopted the symbol $\lrcorner$ (quadrant) in the place of $\frac{\pi}{2}$, a notation which is certainly typical of its meaning.

Thus, $$\lrcorner_\gamma = \int_0^{\lrcorner} \frac{d\varphi}{\Delta\varphi} \quad (= K \text{ of Jacobi}),$$

and the integral to the complementary modulus is

$$\lrcorner_\beta = \int_0^{\lrcorner} \frac{d\varphi}{\sqrt{(1-\beta^2\sin^2\varphi)}} = \int_0^{\lrcorner} \frac{d\varphi}{\Delta(\varphi,\beta)-\beta} \quad (= K' \text{ of Jacobi}).$$

The mode of indicating the modulus with the Δ function, employed here, should in strictness be used altogether. However, it is sufficient to indicate it only for a new modulus.

It is important to notice that if n is an integer, then, and only then,

$$(n\lrcorner)_\gamma = n\lrcorner_\gamma;$$

thus, the integral $$(\tfrac{1}{2}\lrcorner)_\gamma = \int_0^{\frac{1}{2}\lrcorner} \frac{d\varphi}{\Delta\varphi}$$

should not be confounded with $\tfrac{1}{2}\lrcorner_\gamma = \tfrac{1}{2}\int_0^{\lrcorner} \frac{d\varphi}{\Delta\varphi}$.

[TO BE CONTINUED.]

A SIMPLE DISCUSSION OF LOGARITHMIC ERRORS.

By Prof. H. A. Howe, Denver, Col.

[CONTINUED FROM VOL. I, PAGE 130.]

PROBLEM VI.

To determine whether it is more accurate to use the apparent *or the* true *tabular difference.*

Case I. A number is given and the corresponding mantissa required. In Problem I it has been proven that if the apparent tabular difference be employed the resulting error is $xc' + (1 - x)c$. If, now, the true tabular difference be, used, the error of the interpolated mantissa will be (using the nomenclature of Problem II)

$$t + c + x(t' - t) + x(c' - c) - [t + x(t' - t) + x(c' - c)],$$

which reduces to c, as it evidently should. The probable error of c is 0.25 and, if the interpolated mantissa be restricted to n places, another error whose probable value is 0.25, is introduced, so that the final error is $0.25\sqrt{2}$ or 0.35, which is larger than the probable error of the ordinary method, in which the apparent tabular difference is used (see Problem I).

Case II. A mantissa is given and the corresponding natural number is required. If the true tabular difference be used, we divide d by $D + c' - c$ and get

$$\frac{d}{D} - \frac{d}{D^2}c' + \frac{d}{D^2}c + \frac{d}{D^3}(c' - c) + \ldots .$$

Comparing this with the first quotient in Problem II, neglecting terms which contain powers of D higher than the second, and multiples of c and c', we find the error to be $\frac{c}{D}$. Now, the corresponding error of the usual method is found in Problem II to be

$$\frac{d}{D^2}c' + \left(\frac{c}{D} - \frac{d}{D^2}c\right),$$

whose probable value has been shown to be $\frac{0.198}{D}$. But the probable value of $\frac{c}{D}$ is $\frac{0.25}{D}$. We conclude then that in neither case is it advisable to use the true tabular difference.

PROBLEM VII.

To find the probable error of an n-place interpolated mantissa when the corresponding n-place natural number has a probable error of ε in the n^{th} place.

It is plain that we now introduce a new error of $\frac{\varepsilon D}{10^f}$, where f denotes the difference between the number of figures in the tabular argument and the number in the tabular function. Assuming $8(10)^{f-1}$ as the probable value of D and taking 0.31 from Problem I as the average of the probable errors due to other causes, we get for the final result

$$\text{Probable error} = \tfrac{1}{2}\,[0.64\,\varepsilon^2 + (0.31)^2].$$

But if both the magnitude and sign of ε are known, and a particular value of D be given, the actual error due to this cause can be quickly computed.

PROBLEM VIII.

To find the probable error of an n-place interpolated natural number when the corresponding mantissa has a probable error of ε in the n^{th} place.

This error of ε will evidently introduce an error of $\frac{\varepsilon}{D}10^f$, and the total error is seen by Problem II to be

$$\sqrt{\left[\left(\frac{\varepsilon}{D}10^f\right)^2 + \left(\frac{0.198}{D}10^f\right)^2 + (0.25)^2\right]}.$$

If we assume that $8\,(10)^{f-1}$ is the probable value of D, this expression becomes

$$\tfrac{1}{2}\,[1.56\,\varepsilon^2 + (0.35)^2].$$

If the value of D be 10, the effect of the error ε is the same in Problem VII as in Problem VIII; if $D < 10$, the effect is greater in VIII than in VII; and, if $D > 10$, the effect is greater in VII than in VIII.

PROBLEM IX.

Given an n-place table of logarithms of natural numbers, in which the last figure of each mantissa, if too large, is marked with a minus sign: to find the probable error of the mantissa corresponding to a given n-place number, if the mantissa be carried to $n+2$, $n+1$, and n-places respectively.

For simplicity, consider a 5-place table, and apply to the tabular mantissas, just before and after the mantissa to be found, the correction $+0.25$ or -0.25, according as their last figures are unmarked or are marked with a minus sign. Thus each tabular mantissa is carried to seven places and ends with the figure 5. Use the difference between these as the tabular difference, and carry the interpo-

lated mantissa to seven places. Now, the probable value of c or c' is 12.5 units of the seventh place; reasoning as in Problem I, and remembering that no figures are rejected, we find that the probable error of the interpolated mantissa, if $x = 0.1$ is $\frac{1}{2}\sqrt{[(0.1)^2 + (0.9)^2]} \cdot (12.5)$. We then form the following table: —

x	Error.	x	Error.
0.0	12.5	0.5	$(12.5) \cdot \frac{1}{2}\sqrt{(.50)}$
0.1	$(12.5) \cdot \frac{1}{2}\sqrt{(.82)}$	0.6	$(12.5) \cdot \frac{1}{2}\sqrt{(.52)}$
0.2	$(12.5) \cdot \frac{1}{2}\sqrt{(.68)}$	0.7	$(12.5) \cdot \frac{1}{2}\sqrt{(.58)}$
0.3	$(12.5) \cdot \frac{1}{2}\sqrt{(.58)}$	0.8	$(12.5) \cdot \frac{1}{2}\sqrt{(.68)}$
0.4	$(12.5) \cdot \frac{1}{2}\sqrt{(.52)}$	0.9	$(12.5) \cdot \frac{1}{2}\sqrt{(.82)}$

The average of these errors is $(0.813) \cdot (12.5)$, or 10.2 units of the seventh place.

Let us find the probable error if the mantissa be restricted to 6 places. Half the time the tabular differences do not extend beyond the 5th place and half the time they end in 5 of the sixth place. When they do not extend beyond the 5th place and are multiplied by values of x ranging from 0.1 to 0.9, the last figure of the proportional part found affects the 6th place only, and the interpolated mantissa, when carried to seven places, ends in the figure 5. But if the tabular difference end in 5 of the 6th place, the proportional part ends four-ninths of the time in 0 and five-ninths of the time in 5 of the seventh place. So the 7-place interpolated mantissa ends in 0 five-ninths of the time and in 5 four-ninths of the time. Hence in the long run, in 13 cases out of 18, the mantissas end in 5 of the 7th-place and in 5 cases out of 18 they end in 0. Including the case where $x = 0$, so that no interpolation is made, we see that for fifteen-twentieths or three-fourths of the time the resulting mantissa ends in 5 and for one-fourth of the time in 0.

If now the interpolated mantissas be restricted to 6 places, we find that for three-fourths of the time the probable error expressed in units of the 7th-place is $\frac{1}{2}\sqrt{[(10.2)^2 + 5^2]}$, which equals 11.4; for one-fourth of the time the error is 10.2. Combining these, having regard to their weights, we obtain 11.1 as the average of the probable errors. I have taken a series of 90 uniformly distributed numbers and restricted the corresponding mantissa to 6 places. The mean of the resulting errors was 12.1 in the seventh place. Finally, if the mantissa be restricted to 5 places, the probable value of the rejected figures will be 25 units of the seventh place, and since the probable error of the mantissas, when carried to 7 places, is 10.2 units, we have for the present case

$$\begin{aligned} \text{Probable error} &= \tfrac{1}{2}\sqrt{[25^2 + (10.2)^2]} \\ &= 27.0 \text{ units of the seventh place} \\ &= 0.27 \text{ units of the fifth place.} \end{aligned}$$

In the case of the 90 mantissas just mentioned, the average of the errors was

0.25. Comparing the above result with that of Problem I we see that when the interpolated mantissas are restricted to 5 places, the extra labor involved in the use of the table just discussed receives no adequate compensation in increase of accuracy.

It is evident that the maximum error, when the mantissas are carried to 7 places, is 25 units of seventh place; when carried to 6 places, it is 30 units; and when restricted to 5 places, it is 75, or 0.75 of a unit of the fifth place.

PROBLEM X.

Given an n-place mantissa, and the logarithm-table mentioned in Problem IX: to find the probable error of the corresponding number if carried to $n + 1$ and to n places respectively.

We use the notation of Problem II and discuss a 5-place table, remarking that t and t' now denote two 7-place mantissas which end in 25 or 75 and that the probable value of c or c' is 0.125 in units of the fifth place. If we carry the result to 6 places, the formula for the probable resulting error is in units of the sixth place

$$\text{Error} = \sqrt{\left(\left[10\frac{1.25}{D}\sqrt{\left(\frac{d}{D}\right)^2 + \left(1 - \frac{d}{D}\right)^2}\right]^2 + (0.25)^2\right)}.$$

Then, reasoning as in Problem II, the average of the probable errors becomes

$$\sqrt{\left[\left(\frac{9.9}{D}\right)^2 + (0.25)^2\right]}.$$

If the numbers corresponding to the given mantissas be uniformly distributed throughout the table, the average of all the values of the term $\frac{9.9}{D}$ will be $\frac{9.9}{8}$ or 1.24. Substituting this value in the expression just given, we find the result to be 1.26. But if the mantissas be uniformly distributed the average of the values of $\frac{9.9}{D}$ is 0.96, and the average of the probable errors is 0.99.

When the resulting natural numbers are restricted to 5 places, the value of the probable error in units of the fifth place is

$$\sqrt{\left(\left[\frac{1.25}{D}\sqrt{\left(\frac{d}{D}\right)^2 + \left(1 - \frac{d}{D}\right)^2}\right]^2 + (0.25)^2\right)}.$$

Then the average of the probable errors is

$$\sqrt{\left[\left(\frac{0.99}{D}\right)^2 + (0.25)^2\right]}.$$

If the natural numbers be uniformly distributed this reduces to 0.28. When the mantissas are uniformly distributed it becomes 0.27.

Reasoning as in Problem V, we may prove that when the natural numbers are restricted to 5 places, since 4.5 is the smallest tabular difference consistent with the condition that $c = c'$, the maximum error is $\frac{2.5}{4.5} + 0.50$, or 1.06. When the number is carried to 6 places, the maximum error is $10 . \frac{2.5}{4.5} + 0.5$ or 6.1.

I have taken 100 uniformly distributed 5 place mantissas and obtained their antilogarithms from Gernerth's table, carrying the results to 6 places and also to 5 places. More accurate values were then obtained from a 7-place table. Thus it was found that the average of the errors of the 6-place antilogarithms was 1.07 in units of the sixth place instead of 0.99 as the theory gives; the average of the errors of the 5-place antilogarithms was 0.27 in the fifth place, which accords well with the theory.

SOLUTIONS OF EXERCISES.

24

THE curves of the family

$$\left(\frac{C}{r}\right)^n = \cos(p + n\vartheta),$$

where n is a parameter, all pass through a fixed point and cut orthogonally the fixed curve

$$\left(\frac{r}{c}\right)^n = \cos n\vartheta,$$

provided $C^n = \frac{1}{2}c^n \cos p$.

[Generalization of 14.] [*Alfred C. Lane*]

SOLUTION.

When $\vartheta = 0$ we have

$$r = C / \sqrt[n]{(\cos p)} = c / \sqrt[n]{2},$$

which is constant, since c and n are constants. All curves of the family pass, therefore, through the point whose polar co-ordinates are $c / \sqrt[n]{2}$, 0.

At the point of concourse of

$$\left(\frac{C}{r}\right)^n = \cos(p + n\vartheta), \qquad \left(\frac{r}{c}\right)^n = \cos n\vartheta;$$

$$\left(\frac{C}{c}\right)^n = \cos(p + n\vartheta)\cos n\vartheta;$$

$$\therefore \cos p = \cos p + \cos(p + 2n\vartheta);$$

$$\therefore n\vartheta = 45^\circ - \tfrac{1}{2}p.$$

But at this point we have for the first curve

$$\tan\varphi_1 = \frac{rd\vartheta}{dr} = \cot(p + n\vartheta) = \tan n\vartheta$$

and for the second curve

$$\tan\varphi_2 = \frac{rd\vartheta}{dr} = \cot n\vartheta;$$

the two curves, therefore, intersect orthogonally. [*W. M. Thornton.*]

33

THE CUBIC $ax^3 + 3bx^2 + 3cx + d = 0$ has one or three real roots as its discriminant

$$D = (ad - bc)^2 - 4(b^2 - ac)(c^2 - bd)$$

is positive or negative. If D is zero, two roots are in general equal each to

$$\sqrt{\frac{c^2 - bd}{b^2 - ac}}$$

SOLUTION.

Put $v = ax^3 + 3bx^2 + 3cx + d = 0;$

then will
$$v_1 = 3ax^2 + 6bx + 3c,$$
$$v_2 = 2(b^2 - ac)x + bc - ad,$$
$$v_3 = -aD = -a[(ad - bc)^2 - 4(b^2 - ac)(c^2 - bd)].$$

Supposing a, b, c, d to be positive, we shall have when D is positive, provided $b^2 > ac$, the following variations in signs :—

	v	v_1	v_2	v_3	
for $x = -a$	$-$	$+$	$-$	$-$	, two variations,
and for $x = +a$	$+$	$+$	$+$	$-$	, one variation,

under which conditions v has one real root. Suppose the other conditions to remain the same, but D to be negative. In this case, we have

	v	v_1	v_2	v_3	
for $x = -a$	$-$	$+$	$-$	$+$	, three variations,
and for $x = +a$	$+$	$+$	$+$	$+$	; no variations;

and v has, therefore, three real roots.

If v has equal roots, $v_3 = 0$, and, therefore, $D = 0$;

or, $$(ad - bc)^2 - 4(b^2 - ac)(c^2 - bd) = 0;$$

also, the equation $$v^2 = 2(b^2 - ac)x + bc - ad = 0$$

will give for each of the equal roots

$$x = \frac{ad - bc}{2(b^2 - ac)} = \frac{2\sqrt{[(b^2 - ac)(c^2 - bd)]}}{2(b^2 - ac)} = \sqrt{\frac{c^2 - bd}{b^2 - ac}}$$

If $b^2 < ac$, v will have three real roots when D is positive and one when D is negative. [*Asher B. Evans.*]

43

Let O, O' be the centres, r, R the radii of the inscribed and circumscribed circles of a triangle ABC; also let A', B', C' be the feet of the perpendiculars from the vertices on the opposite sides, I their intersection, and ρ the radius of the circle inscribed in $A'B'C'$. Prove

$$OI^2 = 2r^2 - 2R\rho,$$
$$O'I^2 = R^2 - 4R\rho. \qquad [\textit{Wm. E. Heal.}]$$

SOLUTION I.

It is easy to see that AA', BB', CC' are the bisectors of the angles of the triangle $A'B'C'$ and therefore I is the centre of the circle inscribed in $A'B'C'$. Draw IE perpendicular to $A'C'$, OR, $O'R'$ perpendicular to CC', $OS, O'S'$ perpendicular to AB, and OT perpendicular to BC. The triangles $C'IE$, $B'BC$ and $B'CI$, ACC' are similar, and we have

$$C'I : \rho = BC : B'C = a : B'C,$$

$$CI : B'C = AC : C'C = b : \frac{2k}{c} = b : \frac{ab}{2R},$$

where we put $BC = a$, $CA = b$, $AB = c$ and $k =$ area of ABC;

$$\therefore CI \cdot C'I = 2R\rho.$$

We have $$CI = 2O'S' = 2\left\{R^2 - \frac{C^2}{4}\right\}^{\frac{1}{2}} = \frac{R(a^2+b^2-c^2)}{ab},$$

$$CR = CI + C'I - r = \frac{2k}{c} - r = \frac{(a+b+c)r}{c} - r = \frac{(a+b)r}{c},$$

$$CO^2 = OT^2 + CT^2 = r^2 + (s-c)^2 = \frac{4abr^2}{(b+c-a)(c+a-b)},$$

$$C'O^2 - r^2 = CO^2 - CR^2,$$

$$C'O^2 = CO^2 - (CR^2 - r^2) = CO^2 - \frac{(a+b+c)(a+b-c)r^2}{c^2}.$$

1. From the triangle $C'CO$ we have

$$C'I \,.\, CO^2 + CI \,.\, C'O^2 = C'I \,.\, CI^2 + CI \,.\, C'I^2 + (CI + C'I)\, OI^2;$$

whence $$OI^2 = \frac{C'I \,.\, CO^2 + CI \,.\, C'O^2}{CI + C'I} - CI \,.\, C'I$$

$$= \frac{C'I \,.\, CO^2 + CI \,.\, C'O^2}{CI + C'I} - 2R\rho.$$

But $C'I \,.\, CO^2 + CI \,.\, C'O^2$

$$= C'I \,.\, CO^2 + CI \,.\, CO^2 - \frac{(a+b+c)(a+b-c)r^2}{c^2} \,.\, CI,$$

$$CI + C'I = \frac{2k}{c} = \frac{(a+b+c)r}{c};$$

$$\therefore \frac{C'I \,.\, CO^2 + CI \,.\, C'O^2}{CI + C'I} = CO^2 - \frac{(a+b-c)r}{c} \,.\, CI$$

$$= \frac{4abr^2}{(b+c-a)(c+a-b)} - \frac{(a+b-c)(a^2+b^2-c^2)Rr}{abc}$$

$$= 2r^2 + \frac{2(a^2+b^2-c^2)r^2}{(b+c-a)(c+a-b)} - \frac{(a+b-c)(a^2+b^2-c^2)Rr}{abc}.$$

But $$\frac{2(a^2+b^2-c^2)r^2}{(b+c-a)(c+a-b)} = \frac{(a+b-c)(a^2+b^2-c^2)Rr}{abc};$$

$$\therefore OI^2 = 2r^2 - 2R\rho.$$

2. $$O'I^2 = O'C^2 + CI^2 - 2CI \,.\, CR'$$

$$= R^2 + CI^2 - 2CI(CI + C'I - C'R').$$

But $C'R' = O'S' = \frac{1}{2}CI$;

$$\therefore OI^2 = R^2 + CI^2 - 2CI(C'I + \tfrac{1}{2}CI) = R^2 - 2CI \,.\, C'I = R^2 - 4R\rho.$$

Corollary 1. The nine-point circle of ABC is the circumscribed circle $A'B'C'$, its centre N is the middle point of $O'I$, and its radius $R' = \frac{1}{2}R$;

$$\therefore IN^2 = \frac{R^2 - 4R\rho}{4} = R'^2 - 2R'\rho,$$

which is the well known expression for the distance between the centres of the inscribed and circumscribed circles.

Corollary 2. We have

$$O'O^2 + OI^2 = 2IN^2 + 2ON^2;$$

whence
$$ON = \tfrac{1}{2}R - r = R' - r,$$

which shows that the inscribed circle touches the nine-point circle internally.

[*Wm. E. Heal*]

SOLUTION II.

We presuppose an acquaintance with the following well-known propositions:—

1. $OO'^2 = R^2 - 2Rr$;

2. The altitudes of a triangle ABC bisect the angles of $A'B'C'$, thus making I the centre of the circle inscribed in $A'B'C'$;

3. The centre N of the circle about $A'B'C'$ (the "nine-points circle") is on the line $O'I$ and $NO' = NI$;

4. The radius of circle $A'B'C'$ is one-half that of circle ABC;

5. The circle inscribed in the triangle ABC touches the circle $A'B'C'$ internally, making the distance between their centres equal to the difference of their radii.

Thus
$$NI^2 = \left(\frac{R}{2}\right)^2 - 2\left(\frac{R}{2}\right).\rho, \text{ by } 1, 2, 4;$$

i. e.
$$\frac{O'I^2}{4} = \frac{R^2}{4} - R\rho, \text{ by } 3,$$

or
$$O'I^2 = R^2 - 4R\rho.$$

Considering the triangle $OO'I$, N being the middle point of $O'I$,

$$OO'^2 + OI^2 = 2ON^2 + 2NI^2;$$

i. e.
$$R^2 - 2Rr + OI^2 = 2\left(\frac{R}{2} - r\right)^2 + \frac{R^2}{2} - 2R\rho,$$

or
$$OI^2 = 2r^2 - 2R\rho.$$

[*R. D. Bohannan.*]

EXERCISES.

64

A FLY-WHEEL weighing 12000 pounds with a rim of 10 feet diameter is driven by a piston pressure of 3000 pounds acting through a stroke of 3 feet. If the wheel be treated as concentrated in its rim, the weight of all other parts of the mechanism being neglected and the connecting-rod being treated as infinite in length, how many turns will the wheel make in 10 seconds?

[*DeVolson Wood.*]

65

IN a plane triangle ABC perpendiculars to CA, CB at C intercept on AB a segment m. Show how to determine the triangle when m, a, b are given. [In what cases can the triangle be constructed from these data with ruler and compass?—*W. M. T.*] [*Marcus Baker.*]

66

SHOW that the difference between the radii of the inscribed and circumscribed circles of a triangle is a minimum when the triangle is equilateral.

[*R. D. Bohannan.*]

67

A WINS from B on an average 8 games of chess out of 11. What is the value of A's chance in a match for $100 to be given to the player who shall first score three games; drawn games not to be counted? [*L. G. Barbour.*]

68

FROM a point P in the plane of a parabola normals are drawn to the curve. Find the locus of P when the feet of the normals are the corners of a right triangle. [*W. M. Thornton.*]

69

A STRAIGHT LINE is tangent to the parabola $8x - y^2$ and to the circle $x^2 + y^2 - 4$. Required the angle which this line makes with the x-axis.

[*O. Root.*]

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